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A CHARACTERISTIC PROPERTY OF SELF-PROJECTIVE CURVES.

By L. L. Dines.

A plane curve will be said to have the property P_{Ol} with respect to a point O and a line l of the plane, if the pencil of lines obtained by projecting a variable point of the curve from O is projective to the range of points cut out by the tangent at this variable point on l.

Evidently the ∞^3 conics through O and tangent to l have the property thus defined. It is the purpose of this paper to show that the property P_{Ol} characterizes the curves which admit infinitesimal projective transformations* into themselves. These curves have been known by various writers as self-projective curves, anharmonic curves, and W-curves.

The self-projective curves have many interesting properties which have been investigated by Klein,† Lie,‡ and others.§ The characteristic property P_{Ol}, which seems not to have been previously noted, furnishes for the self-projective curves a definition which is in some sense analogous to the Steinerian definition of conics.

1. The Differential Equations of Curves with Property P_{Ol}. In order to determine the curves having the property P_{Ol} (defined in our introduction) with respect to a fixed point O and line l, it is convenient to assume first that O is not on l, and to consider later the excepted case.

Case I. If O is not on l, we choose O as the vertex $(0, 0, 1)$ and l as the side $x_3 = 0$ of the triangle, with reference to which (x_1, x_2, x_3) shall represent the homogeneous coördinates of a point. We are to determine the curves

$$f(x_1, x_2, x_3) = 0 \tag{1}$$

with the property P_{Ol}.

Let P be any point of the curve (1) and let T be the intersection of the tangent at P with l. Then the curve (1) will have the property P_{Ol} if, and only if, the range $[T]$ is projective to the pencil $O[P]$, or what is equivalent, if the pencils $O[T]$ and $O[P]$ are projective. Since the equation of the tangent to (1) at $P(x_1, x_2, x_3)$ is $X_1f_{x_1} + X_2f_{x_2} + X_3f_{x_3} = 0$, the coördinates of its intersection with the line $x_3 = 0$ are $(f_{x_2}, -f_{x_1}, 0)$, from which it follows that the coördinates of the line OT are $(f_{x_1}, f_{x_2}, 0)$. The coördinates of the line OP are easily seen to be $(x_2, -x_1, 0)$. The condition that the pencils $O[T]$ and $O[P]$ be projective is that a homogeneous bilinear relation exist between the coördinates of their corresponding elements. The most general relation of this sort is of the form

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* An infinitesimal projective transformation is equivalent to a one-parameter group of finite projective transformations, and invariance under such a group may be used to define self-projective curves.

† Klein and Lie: "Ueber diejenigen Curven, welche durch ein geschlossenes System von unendlich vielen vertauschbaren Transformationen in sich uebergehen" (Math. Ann., IV, 1871).

‡ Lie-Scheffers: "Vorlesungen ueber continuierliche Gruppen."

§ For a brief treatment of these curves with bibliography, see Loria: "Spezielle algebraische und transcendente ebene Kurven," page 552.

$$(\alpha x_1 + \beta x_2)f_{x_1} + (\gamma x_1 + \delta x_2)f_{x_2} = 0, \tag{2}$$

where α, β, γ, and δ are arbitrary constants. Equation (2) is then the differential equation of the curves with property P_{ol} It is desirable to reduce this equation, as well as others to follow, to non-homogeneous coördinates by means of the relations

$$\frac{x_1}{x_3} = x, \qquad \frac{x_2}{x_3} = y. \tag{3}$$

The equation in non-homogeneous coördinates corresponding to the partial differential equation (2) is then equivalent to the ordinary differential equation

$$\frac{dx}{\alpha x + \beta y} = \frac{dy}{\gamma x + \delta y}. \tag{2'}$$

Since (2′) involves three independent parameters and is of first order, *it represents* ∞^4 *curves.*

Case II. If O is on l, we let l be the side $x_3 = 0$ and O the vertex $(1, 0, 0)$ of the reference triangle, and denote by O' the vertex $(0, 0, 1)$. Then if P and T have the same meanings as in the preceding case, the condition that the range $[T]$ and pencil $O[P]$ be projective is equivalent to the condition that the pencils $O'[T]$ and $O[P]$ be projective, and this latter projectivity finds analytic expression in the bilinear relation

$$(\alpha x_2 + \beta x_3)f_{x_1} + (\gamma x_2 + \delta x_3)f_{x_2} = 0, \tag{4}$$

which upon reduction to non-homogeneous coördinates is equivalent to the ordinary differential equation

$$\frac{dx}{\alpha y + \beta} = \frac{dy}{\gamma y + \delta}. \tag{4'}$$

This equation like (2′) *represents* ∞^4 *curves* which have the property P_{ol}.

2. Concerning Self-Projective Curves. In his theory of infinitesimal transformations, Lie[*] considers the variation of a function due to an infinitesimal projective transformation leaving invariant the line at infinity. Upon reduction to homogeneous coordinates by (3) with the assumption that x_3 remains constant, this variation takes the form

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[*] Lie-Scheffers: "Continuierliche Gruppen"; page 57. The author is indebted to Professor Eisenhart for suggestions regarding the use of Lie's results which led to a considerable shortening of the proof in this section.

$$\delta f \equiv [(\alpha_{11}x_1 + \alpha_{12}x_2 + \alpha_{13}x_3)f_{x_1} + (\alpha_{21}x_1 + \alpha_{22}x_2 + \alpha_{23}x_3)f_{x_2}]\delta t,$$

where δt is an arbitrary infinitesimal, and α_{ij} are constants characterizing the transformation. The curves left invariant by this transformation are the integral curves of the differential equation

$$(\alpha_{11}x_1 + \alpha_{12}x_2 + \alpha_{13}x_3)f_{x_1} + (\alpha_{21}x_1 + \alpha_{22}x_2 + \alpha_{23}x_3)f_{x_2} = 0,$$

and are by definition *self-projective curves.* Since our equations (2) and (4) are of this form we have that *every curve with the property P_{0l} is a self-projective curve.*

Lie has shown* that by suitable choice of a coördinate system, the differential equation representing the self-projective curves can always be reduced to one of the five canonical forms:

$$(5)\qquad (a)\ \frac{dx}{x} = \frac{dy}{ny}; \quad (b)\ \frac{dx}{1} = \frac{dy}{y}; \quad (c)\ \frac{dx}{y} = \frac{dy}{1}; \quad (d)\ \frac{dx}{x} = \frac{dy}{y}; \quad (e)\ \frac{dx}{0} = \frac{dy}{1}.$$

Evidently each of these equations is a special case of either (2′) or (4′). Hence *every self-projective curve has the property P_{0l},* and we have

THEOREM: *The property P_{0l} is characteristic of self-projective curves.*

3. **Types of Curves.** Lie determines† the self-projective curves by integrating the canonical equations (5).

In the present connection it is of interest to examine the various types of curves with reference to the property P_{0l}. For this purpose it is important to recall that the curves (2′) have the property P_{0l} with respect to $O(0, 0, 1)$ and $l(x_3 = 0)$; while the curves (4′) have the property with respect to $O(1, 0, 0)$ and $l(x_3 = 0)$.

Type A. The integration of the canonical equation (*a*) gives a two parameter family of curves, each of which can however by a transformation, which changes only the unit point, be reduced to the form

$$(A)\qquad y = x^n.$$

Since (*a*) is a special case of (2′), each curve of the family (*A*) has the property P_{0l} with respect to the vertex (0, 0, 1) and the opposite side $x_3 = 0$ of the reference triangle.

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* Lie-Scheffers, loc. cit., page 64, Theorem 6; and page 69.

† Lie-Scheffers, loc. cit., pages 69–82.

A somewhat more general result is obtained by writing (A) in homogeneous coördinates in the form

$$x_1^{n_1}x_2^{n_2}x_3^{n_3} = 1,$$

where $n_1 + n_2 + n_3 = 0$. Since any transformation which merely interchanges the sides of the reference triangle transforms this equation into an equation of the same type, we may state more generally that every curve of the family (A) has the property P_{Ol} when O and l are respectively a vertex and opposite side of the reference triangle.

Type A'. The reduction of (2′) to (a) is accomplished by choosing as the reference triangle, the triangle which is invariant under that infinitesimal transformation for which curves (A) are self-projective. The vertex O and side l of this triangle may be supposed to be real. It may happen that the two sides of the invariant triangle which pass through O are imaginary. It is then advantageous to take these imaginary lines as the lines $y = \pm x\sqrt{-1}$. The equations of the curves represented by (A) may then be expressed in polar coördinates in the form

$$(A') \qquad \rho = e^{n'\theta}$$

If l is the line at infinity, the curves (A') are the ordinary logarithmic spirals. That the logarithmic spiral has the property P_{Ol} with respect to the pole and the line at infinity, can be verified directly from the well known property of this curve that it cuts all its radii at a constant angle, which property is equivalent to the equality of the pencils $O[P]$ and $O[T]$.

Type B. The integration of (b) gives a family of curves each of which can be reduced, by a transformation which changes only the unit point, to the form

$$(B) \qquad y = e^x.$$

Since equation (b) is a special case of (4′), the curve (B) has the property P_{Ol} with respect to the point (1, 0, 0) and the line $x_3 = 0$. But the curve (B) also has the property P_{Ol} with respect to the point (0, 1, 0) and the line $x_2 = 0$. For (b) may be transformed into an equation of form (2′) by the transformation $x = x'/y'$, $y = 1/y'$; which changes (0, 1, 0) into (0, 0, 1), and $x_2 = 0$ into $x_3 = 0$.

If, in particular, $x_3 = 0$ is the line at infinity, (B) is the ordinary exponential curve. The exponential curve then has the property P_{Ol} with respect to the point at infinity on the x-axis, and the line at infinity; and also with respect to the point at infinity on the y-axis, and the x-axis.

Type C. Equation (c) is a special case of (4′); hence its integral curves have the property p_{Ol} with respect to the point (1, 0, 0) and the line $x_3 = 0$. Each integral curve can be reduced, by a transformation which does not change the coördinates of O or l, to the form

$$(C) \qquad y^2 = 2x.$$

This is a conic tangent to at O, and the property P_{Ol} is one of the classic properties of conics.

Types D and E. The integration of (d) and (e) gives the straight lines which obviously have the property P_{Ol} with respect to any point and line.

4. Special case of conics. It was noted in our introduction that all conics through O and tangent to l have the property P_{Ol}. If O is not on l, these conics are represented by equation (A) with $n = 2$. If O is on l, they are represented by (C).

Conics may also possess the property P_{Ol} though they neither pass through O nor touch l. These are represented by (A) with $n = -1$ [or (A') with $n' = 0$], the point O being (0, 0, 1) and l being $x_3 = 0$. These are the conics with respect to which l is the polar of O. That such conics have the property P_{Ol} can be verified directly from the fact that the lines OP and OT are in this case conjugate lines with respect to the conic; hence the pencils $O[P]$ and $O[T]$ are projective, the projectivity being in fact involutory.

UNIVERSITY OF SASKATCHEWAN.